

Delta-shocks and vacuums in zero-pressure gas dynamics by the flux approximation ^{*}

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Abstract: In this paper, firstly, by solving the Riemann problem of the zero-pressure flow in gas dynamics with a flux approximation, we construct parameterized delta-shock and constant density solutions, then we show that, as the flux perturbation vanishes, they converge to the delta-shock and vacuum state solutions of the zero-pressure flow, respectively. Secondly, we solve the Riemann problem of the Euler equations of isentropic gas dynamics with a double parameter flux approximation including pressure. Further we rigorously prove that, as the two-parameter flux perturbation vanishes, any Riemann solution containing two shock waves tends to a delta shock solution to the zero-pressure flow; any Riemann solution containing two rarefaction waves tends to a two-contact-discontinuity solution to the zero-pressure flow and the nonvacuum intermediate state in between tends to a vacuum state.

Keywords: Euler equations of isentropic gas dynamics; Zero-pressure flow; Transport equations; Riemann problem; Delta shock wave; Vacuum; flux approximation.

AMS subject classifications: 35L65, 35B30, 76E19, 35Q35, 35L67

1 Introduction

The well-known zero-pressure gas dynamics reads

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2)_x = 0, \end{cases} \quad (1.1)$$

which are also called the transport equations, or Euler equations for pressureless fluids, where ρ is the density and u the velocity. It can be used to model the motion of free particles which stick under collision [6, 2] and the formation of large-scale structures in the universe [15].

In the past twenty years, there has been a great explosion of interests in the extensive investigations on the zero-pressure gas dynamics, for instance, see [1, 6, 2, 17, 13, 12, 7], etc. Among these works, Bouchut [1] first established the existence of measure solutions of the Riemann problem. Weinan E, Rykov and Sinai [6] studied the existence of global weak solution and the behavior of such global solution with random initial data. The 1-D and 2-D Riemann problems were solved by Sheng and Zhang [17] with the characteristic analysis and the vanishing viscosity method, see also [13]. Huang and Wang [7] obtained the uniqueness result of weak solution when the initial data is a Radon measure. In these papers it has been proved that δ -shock waves and vacuum states do occur in solutions. For δ -shock waves, we refer to [9, 8, 10, 19, 20, 21, 23, 24] for more details.

^{*} Supported by the NSF of China (11361073).

During the recent decade, the problem concerning the phenomena of concentration and cavitation and the formation of δ -shock waves and vacuum states in solutions has received much attention. For example, see [3, 4, 11, 27, 18, 22, 5], etc. In 2003, Chen and Liu [3] considered the Euler equations of isentropic gas dynamics

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + P)_x = 0, \end{cases} \quad (1.2)$$

where $\rho \geq 0$, u , P denote the density, the velocity and the pressure respectively. The scalar pressure $P(\rho, \epsilon)$ satisfies

$$\lim_{\epsilon \rightarrow 0} P(\rho, \epsilon) = 0, \quad (1.3)$$

where $\epsilon > 0$ is a small parameter. In their works, in (1.2) Chen and Liu took the prototypical pressure functions for ploytropic gas

$$P(\rho, \epsilon) = \epsilon p(\rho), \quad p(\rho) = \rho^\gamma / \gamma, \quad \gamma > 1. \quad (1.4)$$

They identified and analyzed the phenomena of concentration and cavitation and the formation of δ -shock waves and vacuum states in solutions to the system (1.2) with (1.4) as $\epsilon \rightarrow 0$. Further, in [4] they also studied the nonisentropic fluids. Specially, Li [11] investigated the zero temperature limit for $\gamma = 1$ in (1.4). Besides, the results were extended to the relativistic Euler equations for polytropic gases by Yin and Sheng [27], the perturbed Aw-Rascle model by Shen and Sun [18], etc. Very recently, see [22, 5] for the modified Chaplygin gas pressure law. All in all, these works on this topic are only focused on the pressure level.

Motivated partly by [3, 4, 14], in the present paper, by introducing a flux approximation, we propose to consider the following system

$$\begin{cases} \rho_t + (\rho u - 2\epsilon_1 u)_x = 0, \\ (\rho u)_t + (\rho u^2 - \epsilon_1 u^2 + \epsilon_2 p(\rho))_x = 0, \end{cases} \quad (1.5)$$

where the density $\rho \geq 2\epsilon_1$, $\epsilon_1, \epsilon_2 > 0$ are parameters. Physically, a reasonable perturbation can be used to govern some dynamical behaviors of fluids, so it is worth studying the flux perturbation problem which plays an important role in all the three of theory, application and computation. In contrast to the previous works in [3, 4, 11, 27, 18, 22, 5], we here develop a flux approximation approach which contains the pressure perturbation portion.

Firstly we consider a special case $\epsilon_2 = 0$ in (1.5), that is

$$\begin{cases} \rho_t + (\rho u - 2\epsilon_1 u)_x = 0, \\ (\rho u)_t + (\rho u^2 - \epsilon_1 u^2)_x = 0, \end{cases} \quad (1.6)$$

this is a pure flux approximation of special curiosity. We solve the Riemann problem of the system (1.6) with initial conditions

$$(\rho, u)(0, x) = \begin{cases} (\rho_-, u_-), & x < 0, \\ (\rho_+, u_+), & x > 0, \end{cases} \quad (1.7)$$

where (ρ_{\pm}, u_{\pm}) are arbitrary constants. The Riemann solutions include two kinds of somewhat interesting features. When $u_- < u_+$, the solution consists of two contact discontinuities and a constant density state besides two constant states. When $u_- > u_+$, the solution contains a delta shock wave depending on a parameter. From the solutions constructed, one can find that, compared with the zero-pressure gas dynamics, the vacuum state is removed, while for the δ -shock wave, the location and propagation speed are preserved, the weight decreases. These mean that the flux perturbation works in the pressureless gases.

Then we prove that, as the flux approximation vanishes, that is, parameter $\epsilon_1 \rightarrow 0$, any parameterized delta-shock solution converges to the corresponding one of the zero-pressure flow (1.1). By contrast, any constant density solution goes to the vacuum solution.

Secondly, we solve the Riemann problem (1.5), (1.7). Because both of the characteristic fields are genuinely nonlinear, the elementary waves consist of backward centred rarefaction wave (\overleftarrow{R}), forward centred rarefaction wave (\overrightarrow{R}), backward shock wave (\overleftarrow{S}) and forward shock wave (\overrightarrow{S}). The curves of elementary waves divide the phase plane into five domains. By the analysis method in phase plane, we can establish the existence and uniqueness of Riemann solutions including five different structures.

Moreover, we analyze the limit of Riemann solutions of (1.5) and (1.7) as the double parameter $\epsilon_1, \epsilon_2 \rightarrow 0$. It is shown that when $u_+ < u_-$, the Riemann solution containing two shock waves converges to a delta shock solution, which is exactly the solution to zero-pressure flow (1.1). The density between the two shock waves tends to an extreme concentration in the form of a weighted δ -function, which results in the formation of a delta shock wave. Besides, it is also shown that when $u_+ > u_-$, the Riemann solution containing two rarefaction waves tends to a two-contact-discontinuity solution to zero-pressure flow (1.1), and the nonvacuum intermediate state in between tends to a vacuum state as $\epsilon_1, \epsilon_2 \rightarrow 0$.

Following the above analysis, one can find a fact of interest, that is, the flux approximations of difference have their respective effect on the formation of delta-shock and vacuum state in isentropic fluids. In this regard, it is different from those only in pressure level [3, 4, 11, 27, 18, 22, 5]. Meanwhile, the results obtained show that both the delta shock wave and vacuum are stable under some flux small perturbations. Therefore this work extends in some sense the previous results and proofs in [3, 4, 11]. The flux approximation method can be also extend to the Euler equations for nonisentropic fluids and Chaplygin gas equations [25, 26].

The arrangement of this paper is as follows. In Section 2, we recall the solutions of (1.1), (1.7). Section 3 solves the Riemann problem (1.6), (1.7) and discusses the limits of Riemann solutions. Section 4 solves the Riemann problem for (1.5). Sections 5 and 6 investigate the limit of solutions of (1.5) and (1.7).

2 Delta-shocks and vacuums for the zero-pressure flow

As a start, we briefly recall δ -shocks and vacuum states in the Riemann solutions to the zero-pressure flow (1.1). See [17, 13] for more details.

The system (1.1) has a double eigenvalue $\lambda = u$ with the associated eigenvector $r = (1, 0)^T$ satisfying $\nabla \lambda \cdot r = 0$, which means that it is nonstrictly hyperbolic and λ linearly degenerate.

Consider Riemann problem (1.1), (1.7). By seeking self-similar solution $(\rho, u)(t, x) =$

$(\rho, u)(\xi)$ ($\xi = x/t$), it is easy to find that, besides the constant state and singular solution $\rho = 0, u = \xi$ (vacuum state), the elementary waves of (1.1) are nothing but contact discontinuities. The Riemann problem can be solved by the following two cases.

For the case $u_- < u_+$, the solution includes two contact discontinuities and a vacuum state besides constant states. That is,

$$(\rho, u)(\xi) = \begin{cases} (\rho_-, u_-), & -\infty < \xi < u_-, \\ (0, \xi), & u_- \leq \xi \leq u_+, \\ (\rho_+, u_+), & u_+ < \xi < +\infty. \end{cases} \quad (2.1)$$

For the case $u_- > u_+$, a solution containing a weighted δ -measure (i.e., δ -shock) supported on a line will develop in solutions due to the overlap of characteristic lines.

To define the measure solution, a two-dimensional weighted δ -function $w(s)\delta_S$ supported on a smooth curve S parameterized as $t = t(s)$, $x = x(s)$ ($c \leq s \leq d$) can be defined by

$$\langle w(t(s))\delta_S, \varphi(t(s), x(s)) \rangle = \int_c^d w(t(s))\varphi(t(s), x(s))\sqrt{x'(s)^2 + t'(s)^2}ds \quad (2.2)$$

for all the test functions $\varphi(t, x) \in C_0^\infty(R^+ \times R^1)$.

With this definition, a δ -shock solution of (1.1) can be introduced as follows

$$\rho(t, x) = \rho_0(t, x) + w(t)\delta_S, \quad u(t, x) = u_0(t, x), \quad (2.3)$$

where $S = \{(t, \sigma t) : 0 \leq t < \infty\}$,

$$\rho_0(t, x) = \rho_- + [\rho]\chi(x - \sigma t), u_0(t, x) = u_- + [u]\chi(x - \sigma t), w(t) = \frac{t}{\sqrt{1 + \sigma^2}}(\sigma[\rho] - [\rho u]), \quad (2.4)$$

in which $[g] = g_+ - g_-$, σ is the velocity of the δ -shock, and $\chi(x)$ the characteristic function that is 0 when $x < 0$ and 1 when $x > 0$.

As shown in [17, 13], for any $\varphi(t, x) \in C_0^\infty(R^+ \times R^1)$, the δ -shock solution constructed above satisfies

$$\begin{cases} \langle \rho, \varphi_t \rangle + \langle \rho u, \varphi_x \rangle = 0, \\ \langle \rho u, \varphi_t \rangle + \langle \rho u^2, \varphi_x \rangle = 0, \end{cases} \quad (2.5)$$

where

$$\begin{aligned} \langle \rho, \varphi \rangle &= \int_0^{+\infty} \int_{-\infty}^{+\infty} \rho_0 \varphi dx dt + \langle w\delta_S, \varphi \rangle, \\ \langle \rho u, \varphi \rangle &= \int_0^{+\infty} \int_{-\infty}^{+\infty} \rho_0 u_0 \varphi dx dt + \langle \sigma w\delta_S, \varphi \rangle. \end{aligned} \quad (2.6)$$

Furthermore, substituting (2.3) and (2.4) into (2.5) under the condition (2.2) and (2.6), one can get the generalized Rankine-Hugoniot relation

$$\frac{dx}{dt} = \sigma, \quad \frac{d(w(t)\sqrt{1 + \sigma^2})}{dt} = \sigma[\rho] - [\rho u], \quad \frac{d(w(t)\sigma\sqrt{1 + \sigma^2})}{dt} = \sigma[\rho u] - [\rho u^2] \quad (2.7)$$

which reflects the relationship among the location, weight and propagation speed of the δ -shock wave.

To guarantee the uniqueness, the entropy condition is supplemented as

$$u_+ < \sigma < u_-, \quad (2.8)$$

which means that all characteristic lines on both sides of the discontinuity are not out-going. So it is a overcompressive condition.

Then solving the generalized Rankine-Hugoniot relation (2.7) with initial data $x(0) = 0$ and $w(0) = 0$ under the entropy condition (2.8) yields

$$\sigma = \frac{\sqrt{\rho_+}u_+ + \sqrt{\rho_-}u_-}{\sqrt{\rho_+} + \sqrt{\rho_-}}, \quad w(t) = \frac{\sqrt{\rho_+\rho_-}(u_- - u_+)t}{\sqrt{1 + \sigma^2}}. \quad (2.9)$$

Therefore, a δ -shock solution defined by (2.3) with (2.4) and (2.9) is obtained.

3 Riemann solutions and limit analysis of (1.6) as $\epsilon_1 \rightarrow 0$

The section solves the Riemann problem (1.6), (1.7), and studies the limit of solutions.

For the system (1.6), the eigenvalue and the associated eigenvector are $\lambda = u$ and $r = (1, 0)^T$, respectively, satisfying $\nabla \lambda \cdot r = 0$, which means that the system (1.6) is full linear degenerate and elementary waves only involve contact discontinuities.

In a similar way as the Riemann problem (1.1), (1.7), it is easy to find that, for smooth solutions, besides the constant state, the system (1.6) provides the singular solution

$$\rho = 2\epsilon_1, \quad u = \xi, \quad (3.1)$$

which is called constant density states. While the elementary wave has only contact discontinuity

$$J: \quad \omega = \xi = u_- = u_+, \quad (3.2)$$

which is characterized by $x/t = u_- = u_+$ in (t, x) -plane. It can connect two states (ρ_-, u_-) and (ρ_+, u_+) if and only if they are located on the line $u = u_- = u_+$ in the (ρ, u) -plane.

Now, with constants, constant density state and contact discontinuity, we construct the solutions of Riemann problem (1.6), (1.7) by two cases.

For the case $u_- < u_+$, we draw lines $u = u_-$ and $u = u_+$ from (ρ_-, u_-) and (ρ_+, u_+) , respectively, in the (ρ, u) -plane. These two lines intersect the line $\rho = 2\epsilon_1$ at $(2\epsilon_1, u_-)$ and $(2\epsilon_1, u_+)$. Thus the solution can be constructed by two contact discontinuities and a constant-density state besides two constant states (see Fig. 1), and can be expressed as

$$(\rho, u)(t, x) = (\rho, u)(\xi) = \begin{cases} (\rho_-, u_-), & -\infty < \xi < u_-, \\ (2\epsilon_1, \xi), & u_- \leq \xi \leq u_+, \\ (\rho_+, u_+), & u_+ < \xi < +\infty. \end{cases} \quad (3.3)$$

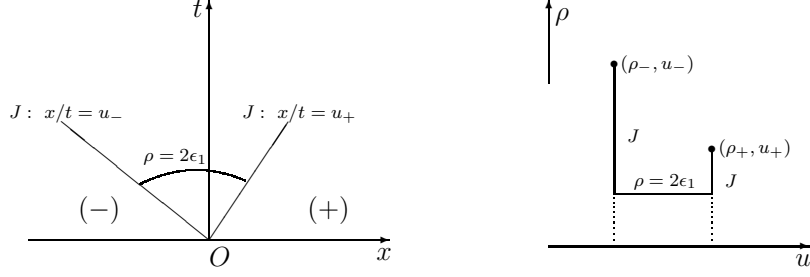


Fig. 1. Constant density.

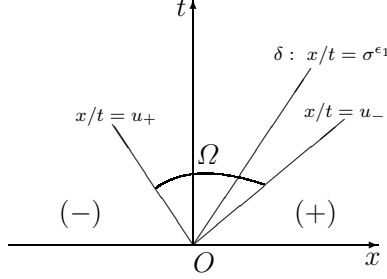


Fig. 2. Characteristic analysis of δ -shock.

For the case $u_- > u_+$, as indicated in Fig. 2, since characteristic lines from initial data overlap each other in the region Ω , so the singularity of solutions must develop in this region. As shown in [17, 13, 21], there is no solutions exist in bounded variation space, then we can construct the Riemann solution by a delta-shock wave.

With the definitions as the above section, we seek a delta shock solution $(\rho^{\epsilon_1}, u^{\epsilon_1}, \sigma^{\epsilon_1}, w^{\epsilon_1})$ of the form (2.3), (2.4), then the following generalized Rankine-Hugoniot relation holds,

$$\begin{cases} \frac{dx}{dt} = \sigma^{\epsilon_1}, \\ \frac{d(w^{\epsilon_1}(t)\sqrt{1+(\sigma^{\epsilon_1})^2})}{dt} = \sigma^{\epsilon_1}[\rho] - [\rho u - 2\epsilon_1 u], \\ \frac{d(w^{\epsilon_1}(t)\sigma^{\epsilon_1}\sqrt{1+(\sigma^{\epsilon_1})^2})}{dt} = \sigma^{\epsilon_1}[\rho u] - [\rho u^2 - \epsilon_1 u^2], \end{cases} \quad (3.4)$$

Besides, the discontinuity should satisfy the entropy condition

$$u_+ < \sigma^{\epsilon_1} < u_-. \quad (3.5)$$

In what follows, the generalized Rankine-Hugoniot relation will be applied in particular to Riemann problem (1.6) and (1.7) for the case $u_- > u_+$. Now this Riemann problem is reduced to solving (3.4) with the initial conditions $t = 0: x(0) = 0, w^{\epsilon_1}(0) = 0$.

Obviously, we have from (3.4) that

$$\begin{cases} w^{\epsilon_1}(t)\sqrt{1+(\sigma^{\epsilon_1})^2} = [\rho]x - [\rho u - 2\epsilon_1 u]t, \\ w^{\epsilon_1}(t)\sigma^{\epsilon_1}\sqrt{1+(\sigma^{\epsilon_1})^2} = [\rho u]x - [\rho u^2 - \epsilon_1 u^2]t. \end{cases} \quad (3.6)$$

Multiplying the first equation by σ^{ϵ_1} and together with the second equation to give

$$\frac{d([\rho]\frac{x^2}{2} - [\rho u]xt)}{dt} = -2[\epsilon_1 u]\sigma^{\epsilon_1}t - [\rho u^2 - \epsilon_1 u^2]t. \quad (3.7)$$

In view of the knowledge concerning delta shock waves in [17, 13, 21], we find that σ^{ϵ_1} is a constant. Then it follows from (3.7) that

$$\frac{[\rho]}{2}x^2 - [\rho u]tx + ([2\epsilon_1 u]\sigma^{\epsilon_1} + [\rho u^2 - \epsilon_1 u^2])\frac{t^2}{2} = 0, \quad (3.8)$$

one solves

$$x(t) = \frac{[\rho u] \pm \sqrt{[\rho u]^2 - [\rho]([2\epsilon_1 u]\sigma^{\epsilon_1} + [\rho u^2 - \epsilon_1 u^2])}}{[\rho]}t, \quad (3.9)$$

as $[\rho] \neq 0$. Then we can obtain

$$\sigma^{\epsilon_1} = \frac{[(\rho - \epsilon_1)u] + \sqrt{(\rho_- - \epsilon_1)(\rho_+ - \epsilon_1)}(u_- - u_+)}{[\rho]}, \quad (3.10)$$

under the entropy condition (3.5). Therefore, from (3.6) we get

$$w^{\epsilon_1}(t) = \frac{[\epsilon_1 u] + \sqrt{(\rho_- - \epsilon_1)(\rho_+ - \epsilon_1)}(u_- - u_+)}{\sqrt{1 + (\sigma^{\epsilon_1})^2}}t. \quad (3.11)$$

Especially, when $[\rho] = 0$,

$$\sigma^{\epsilon_1} = \frac{u_- + u_+}{2}, \quad x(t) = \frac{u_- + u_+}{2}t, \quad w^{\epsilon_1}(t) = \frac{[2\epsilon_1 u - \rho u]}{\sqrt{1 + (\sigma^{\epsilon_1})^2}}t. \quad (3.12)$$

With the above analysis, we reach the following result.

Theorem 3.1. *The Riemann problem (1.6), (1.7) admits a unique weak solution which includes a constant density state as $u_- < u_+$ and a δ -shock wave as $u_- > u_+$.*

In the next, we proceed to discuss the limit of Riemann solutions of the system (1.6) as $\epsilon_1 \rightarrow 0$ for $\rho_- \neq \rho_+$. It needs to investigate two cases: (1) $u_- > u_+$, (2) $u_- < u_+$.

We first consider Case (1), which is relevant to the formation of δ -shock waves.

Computing the limits of σ^{ϵ_1} and $w^{\epsilon_1}(t)$ as $\epsilon_1 \rightarrow 0$, from (3.10) and (3.11), one can obtain

$$\lim_{\epsilon_1 \rightarrow 0} \sigma^{\epsilon_1} = \frac{\sqrt{\rho_+}u_+ + \sqrt{\rho_-}u_-}{\sqrt{\rho_+} + \sqrt{\rho_-}} = \sigma, \quad \lim_{\epsilon_1 \rightarrow 0} w^{\epsilon_1}(t) = \frac{\sqrt{\rho_+ \rho_-}(u_- - u_+)}{\sqrt{1 + \sigma^2}}t = w(t). \quad (3.13)$$

In a simple way similar to that in [3, 11], one can easily conclude the following results.

Theorem 3.2. *Let $u_- > u_+$. For each fixed $\epsilon_1 > 0$, assume that $(\rho^{\epsilon_1}, u^{\epsilon_1})$ is a δ -shock solution of (1.6), (1.7). Then, when $\epsilon_1 \rightarrow 0$, the pair of limit functions (ρ, u) is a δ -shock solution of (1.1), (1.7). Moreover, ρ and ρu are the sum of a step function and a δ -measure with weights $\frac{t}{\sqrt{1 + \sigma^2}}(\sigma[\rho] - [\rho u])$ and $\frac{t}{\sqrt{1 + \sigma^2}}(\sigma[\rho u] - [\rho u^2])$, respectively.*

Now we turn to Case (2). At this moment, the solution of Riemann problem (1.6), (1.7) can be expressed as (3.3). It is obvious to get that, as $\epsilon_1 \rightarrow 0$, the limit of solution to the system (1.6) is just the vacuum solution (2.1) to the zero-pressure flow.

4 Solutions of Riemann problem (1.5) and (1.7)

In this section, we solve the elementary waves and construct the solutions of Riemann problem (1.5) and (1.7). For any $\epsilon_1, \epsilon_2 > 0$, the system (1.5) has two eigenvalues

$$\lambda_1 = u - \sqrt{\epsilon_2 \rho^{\gamma-2}(\rho - 2\epsilon_1)}, \quad \lambda_2 = u + \sqrt{\epsilon_2 \rho^{\gamma-2}(\rho - 2\epsilon_1)}, \quad (4.1)$$

so it is strictly hyperbolic. The corresponding right eigenvectors are

$$r_1 = \left(1, -\sqrt{\frac{\epsilon_2 \rho^{\gamma-2}}{\rho - 2\epsilon_1}}\right)^T, \quad r_2 = \left(1, \sqrt{\frac{\epsilon_2 \rho^{\gamma-2}}{\rho - 2\epsilon_1}}\right)^T.$$

Since $\nabla \lambda_i \cdot r_i \neq 0$ ($i = 1, 2$), both of the characteristic fields are genuinely nonlinear.

Seeking the self-similar solution, we reach the following boundary value problem

$$\begin{cases} -\xi \rho_\xi + (\rho u - 2\epsilon_1 u)_\xi = 0, \\ -\xi(\rho u)_\xi + (\rho u^2 - \epsilon_1 u^2 + \frac{\epsilon_2 \rho^\gamma}{\gamma})_\xi = 0, \end{cases} \quad (4.2)$$

and

$$(\rho, u)(\pm\infty) = (\rho_\pm, u_\pm). \quad (4.3)$$

For any smooth solution, (4.2) is equivalent to

$$\begin{pmatrix} -\xi + u & \rho - 2\epsilon_1 \\ -\xi u + u^2 + \epsilon_2 \rho^{\gamma-1} & -\xi \rho + 2\rho u - 2\epsilon_1 u \end{pmatrix} \begin{pmatrix} d\rho \\ du \end{pmatrix} = 0, \quad (4.4)$$

which provides either the general solution (constant state)

$$(\rho, u)(\xi) = \text{constant}, \quad (4.5)$$

or the backward centred rarefaction wave

$$\overleftarrow{R}(\rho_-, u_-) : \begin{cases} \xi = \lambda_1 = u - \sqrt{\epsilon_2 \rho^{\gamma-2}(\rho - 2\epsilon_1)}, \\ u - u_- = -\int_{\rho_-}^{\rho} \sqrt{\frac{\epsilon_2 s^{\gamma-2}}{s - 2\epsilon_1}} ds, \end{cases} \quad \rho < \rho_-, \quad (4.6)$$

or the forward centred rarefaction wave

$$\overrightarrow{R}(\rho_-, u_-) : \begin{cases} \xi = \lambda_2 = u + \sqrt{\epsilon_2 \rho^{\gamma-2}(\rho - 2\epsilon_1)}, \\ u - u_- = \int_{\rho_-}^{\rho} \sqrt{\frac{\epsilon_2 s^{\gamma-2}}{s - 2\epsilon_1}} ds, \end{cases} \quad \rho > \rho_-. \quad (4.7)$$

For the backward centred rarefaction wave, differentiating u with respect to ρ in the second equation of (4.6), it follows that $u_\rho = -\sqrt{\frac{\epsilon_2 \rho^{\gamma-2}}{\rho - 2\epsilon_1}} < 0$. For the forward centred rarefaction wave, it is easy to see that $u_\rho = \sqrt{\frac{\epsilon_2 \rho^{\gamma-2}}{\rho - 2\epsilon_1}} > 0$.

Taking the limit $\rho \rightarrow 2\epsilon_1$ in the second equation of (4.6) leads to

$$\lim_{\rho \rightarrow 2\epsilon_1} u = u_- + \int_{2\epsilon_1}^{\rho_-} \sqrt{\frac{\epsilon_2 s^{\gamma-2}}{s - 2\epsilon_1}} ds. \quad (4.8)$$

Since $\lim_{s \rightarrow 2\epsilon_1} \left((s - 2\epsilon_1)^{\frac{1}{2}} \sqrt{\frac{\epsilon_2 s^{\gamma-2}}{s - 2\epsilon_1}} \right) = \sqrt{\epsilon_2 (2\epsilon_1)^{\gamma-2}}$, the integral $\int_{2\epsilon_1}^{\rho_-} \sqrt{\frac{\epsilon_2 s^{\gamma-2}}{s - 2\epsilon_1}} ds$ is convergent due to Cauchy criterion. Thus, from (4.8), we can conclude that the backward

centred rarefaction wave curve intersects with the line $\rho = 2\epsilon_1$ at the point $(2\epsilon_1, u_1) = (2\epsilon_1, u_- + \int_{2\epsilon_1}^{\rho_-} \sqrt{\frac{\epsilon_2 s^{\gamma-2}}{s - 2\epsilon_1}} ds)$.

Performing the limit $\rho \rightarrow +\infty$ in the second equation in (4.7) yields

$$\lim_{\rho \rightarrow +\infty} u = u_- + \int_{\rho_-}^{+\infty} \sqrt{\frac{\epsilon_2 s^{\gamma-2}}{s - 2\epsilon_1}} ds. \quad (4.9)$$

Since $\sqrt{\frac{\epsilon_2 s^{\gamma-2}}{s - 2\epsilon_1}} > \sqrt{\frac{\epsilon_2 s^{\gamma-2}}{s}}$, we have

$$\int_{\rho_-}^{+\infty} \sqrt{\frac{\epsilon_2 s^{\gamma-2}}{s - 2\epsilon_1}} ds > \int_{\rho_-}^{+\infty} \sqrt{\frac{\epsilon_2 s^{\gamma-2}}{s}} ds = +\infty. \quad (4.10)$$

Thus, from (4.9), one deduces that $\lim_{\rho \rightarrow +\infty} u = +\infty$.

For a bounded discontinuity at $\xi = \sigma^{\epsilon_1 \epsilon_2}$, the Rankine-Hugoniot relation

$$\begin{cases} -\sigma^{\epsilon_1 \epsilon_2} [\rho] + [\rho u - 2\epsilon_1 u] = 0, \\ -\sigma^{\epsilon_1 \epsilon_2} [\rho u] + [\rho u^2 - \epsilon_1 u^2 + \frac{\epsilon_2 \rho^\gamma}{\gamma}] = 0, \end{cases} \quad (4.11)$$

holds, where $[q] = q_r - q_l$ with $q_l = q(t, x(t) - 0)$ and $q_r = q(t, x(t) + 0)$.

Eliminating $\sigma^{\epsilon_1 \epsilon_2}$ from (4.11), we get

$$(\rho_l \rho_r - \epsilon_1 (\rho_l + \rho_r))(u_r - u_l)^2 = \frac{\epsilon_2}{\gamma} (\rho_r - \rho_l)(\rho_r^\gamma - \rho_l^\gamma), \quad (4.12)$$

which yields $\rho_l \rho_r - \epsilon_1 (\rho_l + \rho_r) > 0$. Thus, we have

$$u_r - u_l = \pm \sqrt{\frac{\epsilon_2 (\rho_r - \rho_l)(\rho_r^\gamma - \rho_l^\gamma)}{\gamma (\rho_l \rho_r - \epsilon_1 (\rho_l + \rho_r))}}. \quad (4.13)$$

Using the Lax entropy inequalities, one can get that the backward shock wave satisfies

$$\sigma^{\epsilon_1 \epsilon_2} < \lambda_1(\rho_l, u_l), \quad \lambda_1(\rho_r, u_r) < \sigma^{\epsilon_1 \epsilon_2} < \lambda_2(\rho_r, u_r), \quad (4.14)$$

and the forward shock wave satisfies

$$\lambda_1(\rho_l, u_l) < \sigma^{\epsilon_1 \epsilon_2} < \lambda_2(\rho_l, u_l), \quad \lambda_2(\rho_r, u_r) < \sigma^{\epsilon_1 \epsilon_2}. \quad (4.15)$$

Then we can obtain that the following inequality holds for the backward shock wave

$$\frac{-\sqrt{\epsilon_2 \rho_r^{\gamma-2}(\rho_r - 2\epsilon_1)}}{\rho_l - 2\epsilon_1} < \frac{u_r - u_l}{\rho_r - \rho_l} < \frac{-\sqrt{\epsilon_2 \rho_l^{\gamma-2}(\rho_l - 2\epsilon_1)}}{\rho_r - 2\epsilon_1}, \quad (4.16)$$

which implies that $\rho_l < \rho_r$ and $u_r < u_l$.

In an analogous way, for the forward shock wave, we have

$$\frac{\sqrt{\epsilon_2 \rho_r^{\gamma-2}(\rho_r - 2\epsilon_1)}}{\rho_l - 2\epsilon_1} < \frac{u_r - u_l}{\rho_r - \rho_l} < \frac{\sqrt{\epsilon_2 \rho_l^{\gamma-2}(\rho_l - 2\epsilon_1)}}{\rho_r - 2\epsilon_1}, \quad (4.17)$$

which gives $\rho_l > \rho_r$ and $u_r < u_l$.

Thus, given a left state (ρ_-, u_-) , one can get the backward shock wave curve

$$\overleftarrow{S}(\rho_-, u_-) : u - u_- = -\sqrt{\frac{\epsilon_2(\rho - \rho_-)(\rho^\gamma - \rho_-^\gamma)}{\gamma(\rho - \rho_- - \epsilon_1(\rho + \rho_-))}}, \quad \rho > \rho_-, \quad (4.18)$$

and the forward shock wave curve

$$\overrightarrow{S}(\rho_-, u_-) : u - u_- = -\sqrt{\frac{\epsilon_2(\rho - \rho_-)(\rho^\gamma - \rho_-^\gamma)}{\gamma(\rho - \rho_- - \epsilon_1(\rho + \rho_-))}}, \quad \rho < \rho_-. \quad (4.19)$$

In addition, for the backward shock wave, differentiating u with respect to ρ in (4.18), it is immediate that

$$u_\rho = -\frac{\epsilon_2}{2} \left(\frac{\epsilon_2(\rho - \rho_-)(\rho^\gamma - \rho_-^\gamma)}{\gamma(\rho - \rho_- - \epsilon_1(\rho + \rho_-))} \right)^{-\frac{1}{2}} \frac{I}{\gamma(\rho - \rho_- - \epsilon_1(\rho + \rho_-))^2} < 0, \quad (4.20)$$

where $I = \rho_-(\rho_- - 2\epsilon_1)(\rho^\gamma - \rho_-^\gamma) + \gamma\rho^{\gamma-1}(\rho - \rho_- - \epsilon_1(\rho + \rho_-))(\rho - \rho_-)$. Similarly, for the forward shock wave, we have $u_\rho > 0$.

When $\rho \rightarrow +\infty$ in (4.18), we find $\lim_{\rho \rightarrow +\infty} u = -\infty$. When $\rho \rightarrow 2\epsilon_1$ in (4.19), we obtain

$$\lim_{\rho \rightarrow 2\epsilon_1} u = u_- - \sqrt{\frac{\epsilon_2(\rho_-^\gamma - (2\epsilon_1)^\gamma)}{\gamma\epsilon_1}}, \quad (4.21)$$

which shows that the forward shock wave curve intersects with the line $\rho = 2\epsilon_1$ at the point $(2\epsilon_1, u_2) = (2\epsilon_1, u_- - \sqrt{\frac{\epsilon_2(\rho_-^\gamma - (2\epsilon_1)^\gamma)}{\gamma\epsilon_1}})$.

Through the analysis above, as illustrated in Fig. 3, fixing a left state (ρ_-, u_-) , the phase plane can be divided into five regions by the wave curves.

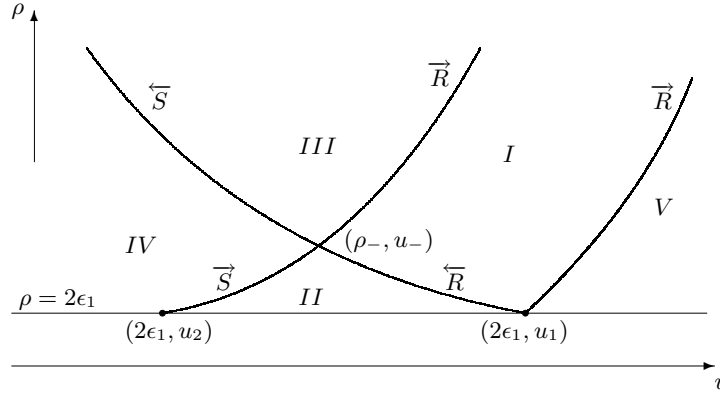


Fig. 3. Curves of elementary waves.

Now, according to the right state (ρ_+, u_+) in the different regions, one can get five kinds of configurations of solutions. Particularly, when $(\rho_+, u_+) \in \overleftarrow{S} \overrightarrow{S}(\rho_-, u_-)$, the Riemann solution contains two shock waves and a nonvacuum intermediate constant states whose density may become singular as $\epsilon_1, \epsilon_2 \rightarrow 0$. When $(\rho_+, u_+) \in \overleftarrow{R} \overrightarrow{R}(\rho_-, u_-)$, the Riemann solution contains two rarefaction waves and a intermediate state that may be a constant density solution ($\rho = 2\epsilon_1$). Since the other two regions $\overleftarrow{S} \overrightarrow{R}(\rho_-, u_-)$ and $\overleftarrow{R} \overrightarrow{S}(\rho_-, u_-)$ have empty interiors when $\epsilon_1, \epsilon_2 \rightarrow 0$, it suffices to study the limit process for the two cases $(\rho_+, u_+) \in \overleftarrow{S} \overrightarrow{S}(\rho_-, u_-)$ and $(\rho_+, u_+) \in \overleftarrow{R} \overrightarrow{R}(\rho_-, u_-)$.

5 Formation of delta shock waves for the system (1.5)

This section analyzes the limit as $\epsilon_1, \epsilon_2 \rightarrow 0$ of solutions of (1.5) and (1.7) in the case $(\rho_+, u_+) \in \overleftarrow{S} \overrightarrow{S}(\rho_-, u_-)$ with $u_- > u_+$.

5.1. Limit behavior of the Riemann solutions as $\epsilon_1, \epsilon_2 \rightarrow 0$

For any $\epsilon_1, \epsilon_2 > 0$, let $(\rho_*^{\epsilon_1 \epsilon_2}, u_*^{\epsilon_1 \epsilon_2})$ be the intermediate state in the sense that (ρ_-, u_-) and $(\rho_*^{\epsilon_1 \epsilon_2}, u_*^{\epsilon_1 \epsilon_2})$ are connected by backward shock wave \overleftarrow{S} with speed $\sigma_1^{\epsilon_1 \epsilon_2}$ and that $(\rho_*^{\epsilon_1 \epsilon_2}, u_*^{\epsilon_1 \epsilon_2})$ and (ρ_+, u_+) are connected by forward shock wave \overrightarrow{S} with speed $\sigma_2^{\epsilon_1 \epsilon_2}$. They have the following relations

$$u_*^{\epsilon_1 \epsilon_2} - u_- = -\sqrt{\frac{\epsilon_2(\rho_*^{\epsilon_1 \epsilon_2} - \rho_-)((\rho_*^{\epsilon_1 \epsilon_2})^\gamma - \rho_-^\gamma)}{\gamma(\rho_- \rho_*^{\epsilon_1 \epsilon_2} - \epsilon_1(\rho_- + \rho_*^{\epsilon_1 \epsilon_2}))}}, \quad \rho_*^{\epsilon_1 \epsilon_2} > \rho_- \quad (5.1)$$

on \overleftarrow{S} , and

$$u_+ - u_*^{\epsilon_1 \epsilon_2} = -\sqrt{\frac{\epsilon_2(\rho_+ - \rho_*^{\epsilon_1 \epsilon_2})(\rho_+^\gamma - (\rho_*^{\epsilon_1 \epsilon_2})^\gamma)}{\gamma(\rho_*^{\epsilon_1 \epsilon_2} \rho_+ - \epsilon_1(\rho_*^{\epsilon_1 \epsilon_2} + \rho_+))}}, \quad \rho_*^{\epsilon_1 \epsilon_2} > \rho_+ \quad (5.2)$$

on \overrightarrow{S} . Then we have the following lemmas.

Lemma 5.1. $\lim_{\epsilon_1, \epsilon_2 \rightarrow 0} \rho_*^{\epsilon_1 \epsilon_2} = +\infty$.

Proof. Suppose that $\lim_{\epsilon_1, \epsilon_2 \rightarrow 0} \rho_*^{\epsilon_1 \epsilon_2} = M \in (\max(\rho_-, \rho_+), +\infty)$. It follows from (5.1) and (5.2) that

$$u_+ - u_- = -\sqrt{\frac{\epsilon_2}{\gamma}} \left(\sqrt{\frac{(\rho_*^{\epsilon_1 \epsilon_2} - \rho_-)((\rho_*^{\epsilon_1 \epsilon_2})^\gamma - \rho_-^\gamma)}{\rho_- \rho_*^{\epsilon_1 \epsilon_2} - \epsilon_1(\rho_- + \rho_*^{\epsilon_1 \epsilon_2})}} + \sqrt{\frac{(\rho_+ - \rho_*^{\epsilon_1 \epsilon_2})(\rho_+^\gamma - (\rho_*^{\epsilon_1 \epsilon_2})^\gamma)}{\rho_*^{\epsilon_1 \epsilon_2} \rho_+ - \epsilon_1(\rho_*^{\epsilon_1 \epsilon_2} + \rho_+)}} \right). \quad (5.3)$$

Letting $\epsilon_1, \epsilon_2 \rightarrow 0$ in (5.3), one can get $u_+ = u_-$, which contradicts $u_+ < u_-$. Therefore, Lemma 5.1 holds. \blacksquare

Letting $\epsilon_1, \epsilon_2 \rightarrow 0$ in (5.3), one can directly get

Lemma 5.2. $\lim_{\epsilon_1, \epsilon_2 \rightarrow 0} \epsilon_2(\rho_*^{\epsilon_1 \epsilon_2})^\gamma = \gamma \rho_- \rho_+ \left(\frac{u_- - u_+}{\sqrt{\rho_-} + \sqrt{\rho_+}} \right)^2$.

Lemma 5.3. Set $\sigma = \frac{\sqrt{\rho_-} u_- + \sqrt{\rho_+} u_+}{\sqrt{\rho_-} + \sqrt{\rho_+}}$. Then

$$\lim_{\epsilon_1, \epsilon_2 \rightarrow 0} u_*^{\epsilon_1 \epsilon_2} = \lim_{\epsilon_1, \epsilon_2 \rightarrow 0} \sigma_1^{\epsilon_1 \epsilon_2} = \lim_{\epsilon_1, \epsilon_2 \rightarrow 0} \sigma_2^{\epsilon_1 \epsilon_2} = \sigma. \quad (5.4)$$

Proof. Passing to the limit $\epsilon_1, \epsilon_2 \rightarrow 0$ in (5.1) and noticing Lemma 5.2, we have

$$\lim_{\epsilon_1, \epsilon_2 \rightarrow 0} u_*^{\epsilon_1 \epsilon_2} = u_- - \frac{1}{\sqrt{\gamma \rho_-}} \lim_{\epsilon_1, \epsilon_2 \rightarrow 0} \sqrt{\epsilon_2(\rho_*^{\epsilon_1 \epsilon_2})^\gamma} = \sigma. \quad (5.5)$$

Form (4.11), $\sigma_1^{\epsilon_1 \epsilon_2}$ and $\sigma_2^{\epsilon_1 \epsilon_2}$ can be calculated by

$$\sigma_1^{\epsilon_1 \epsilon_2} = u_*^{\epsilon_1 \epsilon_2} + \frac{(\rho_- - 2\epsilon_1)(u_*^{\epsilon_1 \epsilon_2} - u_-)}{\rho_*^{\epsilon_1 \epsilon_2} - \rho_-}, \quad \sigma_2^{\epsilon_1 \epsilon_2} = u_*^{\epsilon_1 \epsilon_2} + \frac{(\rho_+ - 2\epsilon_1)(u_+ - u_*^{\epsilon_1 \epsilon_2})}{\rho_+ - \rho_*^{\epsilon_1 \epsilon_2}}, \quad (5.6)$$

thus, $\lim_{\epsilon_1, \epsilon_2 \rightarrow 0} \sigma_1^{\epsilon_1 \epsilon_2} = \lim_{\epsilon_1, \epsilon_2 \rightarrow 0} \sigma_2^{\epsilon_1 \epsilon_2} = \lim_{\epsilon_1, \epsilon_2 \rightarrow 0} u_*^{\epsilon_1 \epsilon_2}$. So the lemma is true. \blacksquare

Lemma 5.1 and Lemma 5.3 show that when ϵ_1 and ϵ_2 drop to zero, \overleftarrow{S} and \overrightarrow{S} coincide, the intermediate density $\rho_*^{\epsilon_1 \epsilon_2}$ becomes singular.

Combining (5.6) with Lemma 5.1 and Lemma 5.3, we have the following result.

Lemma 5.4. $\lim_{\epsilon_1, \epsilon_2 \rightarrow 0} \rho_*^{\epsilon_1 \epsilon_2} (\sigma_2^{\epsilon_1 \epsilon_2} - \sigma_1^{\epsilon_1 \epsilon_2}) = \sigma[\rho] - [\rho u]$.

5.2. Weighted delta shock waves

Now, we show the theorem characterizing the limit as $\epsilon_1, \epsilon_2 \rightarrow 0$ for the case $u_+ < u_-$ and $(\rho_+, u_+) \in \overleftarrow{S} \overrightarrow{S}(\rho_-, u_-)$.

Theorem 5.5. *Let $u_+ < u_-$. Assume $(\rho^{\epsilon_1 \epsilon_2}, u^{\epsilon_1 \epsilon_2})$ is a two-shock wave solution of (1.5) and (1.7) constructed in Section 4. Then, when $\epsilon_1, \epsilon_2 \rightarrow 0$, $\rho^{\epsilon_1 \epsilon_2}$ and $\rho^{\epsilon_1 \epsilon_2} u^{\epsilon_1 \epsilon_2}$ converge in the sense of distributions, and the limit functions of $\rho^{\epsilon_1 \epsilon_2}$ and $\rho^{\epsilon_1 \epsilon_2} u^{\epsilon_1 \epsilon_2}$ are the sum of a step function and a δ -function with the weights*

$$\frac{t}{\sqrt{1+\sigma^2}}(\sigma[\rho] - [\rho u]) \quad \text{and} \quad \frac{t}{\sqrt{1+\sigma^2}}(\sigma[\rho u] - [\rho u^2]),$$

respectively, which form a delta shock solution of (1.1) with the Riemann data (1.7).

Proof. (i). Set $\xi = x/t$. Then, for each $\epsilon_1, \epsilon_2 > 0$, the Riemann solution containing two shocks can be expressed as

$$(\rho^{\epsilon_1 \epsilon_2}, u^{\epsilon_1 \epsilon_2})(\xi) = \begin{cases} (\rho_-, u_-), & \xi < \sigma_1^{\epsilon_1 \epsilon_2}, \\ (\rho_*^{\epsilon_1 \epsilon_2}, u_*^{\epsilon_1 \epsilon_2}), & \sigma_1^{\epsilon_1 \epsilon_2} < \xi < \sigma_2^{\epsilon_1 \epsilon_2}, \\ (\rho_+, u_+), & \xi > \sigma_2^{\epsilon_1 \epsilon_2}, \end{cases} \quad (5.7)$$

satisfying weak formulations: For any $\phi \in C_0^1(-\infty, +\infty)$,

$$\int_{-\infty}^{+\infty} (\rho^{\epsilon_1 \epsilon_2} u^{\epsilon_1 \epsilon_2} - \rho^{\epsilon_1 \epsilon_2} \xi - 2\epsilon_1 u^{\epsilon_1 \epsilon_2}) \phi' d\xi - \int_{-\infty}^{+\infty} \rho^{\epsilon_1 \epsilon_2} \phi d\xi = 0, \quad (5.8)$$

and

$$\int_{-\infty}^{+\infty} \left((\rho^{\epsilon_1 \epsilon_2} - \epsilon_1)(u^{\epsilon_1 \epsilon_2})^2 + \frac{\epsilon_2 (\rho^{\epsilon_1 \epsilon_2})^\gamma}{\gamma} - \rho^{\epsilon_1 \epsilon_2} u^{\epsilon_1 \epsilon_2} \xi \right) \phi' d\xi - \int_{-\infty}^{+\infty} \rho^{\epsilon_1 \epsilon_2} u^{\epsilon_1 \epsilon_2} \phi d\xi = 0. \quad (5.9)$$

(ii). The first integral in (5.8) can be decomposed into

$$\left(\int_{-\infty}^{\sigma_1^{\epsilon_1 \epsilon_2}} + \int_{\sigma_1^{\epsilon_1 \epsilon_2}}^{\sigma_2^{\epsilon_1 \epsilon_2}} + \int_{\sigma_2^{\epsilon_1 \epsilon_2}}^{+\infty} \right) (\rho^{\epsilon_1 \epsilon_2} u^{\epsilon_1 \epsilon_2} - \rho^{\epsilon_1 \epsilon_2} \xi - 2\epsilon_1 u^{\epsilon_1 \epsilon_2}) \phi' d\xi. \quad (5.10)$$

The limit of the sum of the first and last term of (5.10) equals

$$\begin{aligned} & \lim_{\epsilon_1, \epsilon_2 \rightarrow 0} \int_{-\infty}^{\sigma_1^{\epsilon_1 \epsilon_2}} (\rho_- u_- - \rho_- \xi - 2\epsilon_1 u_-) \phi' d\xi + \lim_{\epsilon_1, \epsilon_2 \rightarrow 0} \int_{\sigma_2^{\epsilon_1 \epsilon_2}}^{+\infty} (\rho_+ u_+ - \rho_+ \xi - 2\epsilon_1 u_+) \phi' d\xi \\ &= (\sigma[\rho] - [\rho u]) \phi(\sigma) + \int_{-\infty}^{+\infty} H(\xi - \sigma) \phi d\xi \end{aligned} \quad (5.11)$$

with $H(\xi - \sigma)$ taking ρ_- for $\xi < \sigma$ and ρ_+ for $\xi > \sigma$, respectively. While the limit of the second term of (5.10) can be written as

$$\begin{aligned}
& \lim_{\epsilon_1, \epsilon_2 \rightarrow 0} \int_{\sigma_1^{\epsilon_1 \epsilon_2}}^{\sigma_2^{\epsilon_1 \epsilon_2}} (\rho_*^{\epsilon_1 \epsilon_2} u_*^{\epsilon_1 \epsilon_2} - \rho_*^{\epsilon_1 \epsilon_2} \xi - 2\epsilon_1 u_*^{\epsilon_1 \epsilon_2}) \phi' d\xi \\
&= \lim_{\epsilon_1, \epsilon_2 \rightarrow 0} \rho_*^{\epsilon_1 \epsilon_2} (\sigma_2^{\epsilon_1 \epsilon_2} - \sigma_1^{\epsilon_1 \epsilon_2}) \left(\frac{\phi(\sigma_2^{\epsilon_1 \epsilon_2}) - \phi(\sigma_1^{\epsilon_1 \epsilon_2})}{\sigma_2^{\epsilon_1 \epsilon_2} - \sigma_1^{\epsilon_1 \epsilon_2}} u_*^{\epsilon_1 \epsilon_2} - \frac{\sigma_2^{\epsilon_1 \epsilon_2} \phi(\sigma_2^{\epsilon_1 \epsilon_2}) - \sigma_1^{\epsilon_1 \epsilon_2} \phi(\sigma_1^{\epsilon_1 \epsilon_2})}{\sigma_2^{\epsilon_1 \epsilon_2} - \sigma_1^{\epsilon_1 \epsilon_2}} \right. \\
&\quad \left. + \frac{1}{\sigma_2^{\epsilon_1 \epsilon_2} - \sigma_1^{\epsilon_1 \epsilon_2}} \int_{\sigma_1^{\epsilon_1 \epsilon_2}}^{\sigma_2^{\epsilon_1 \epsilon_2}} \phi d\xi \right) - \lim_{\epsilon_1, \epsilon_2 \rightarrow 0} 2\epsilon_1 u_*^{\epsilon_1 \epsilon_2} \left(\phi(\sigma_2^{\epsilon_1 \epsilon_2}) - \phi(\sigma_1^{\epsilon_1 \epsilon_2}) \right) \\
&= (\sigma[\rho] - [\rho u]) \left(\sigma \phi'(\sigma) - \sigma \phi'(\sigma) - \phi(\sigma) + \phi(\sigma) \right) \\
&= 0.
\end{aligned} \tag{5.12}$$

Returning to (5.8), we immediately obtain that

$$\lim_{\epsilon_1, \epsilon_2 \rightarrow 0} \int_{-\infty}^{+\infty} \rho^{\epsilon_1 \epsilon_2} \phi d\xi = (\sigma[\rho] - [\rho u]) \phi(\sigma) + \int_{-\infty}^{+\infty} H(\xi - \sigma) \phi d\xi. \tag{5.13}$$

(iii) Now we consider the limit of $\rho^{\epsilon_1 \epsilon_2} u^{\epsilon_1 \epsilon_2}$. In the same way as before, we decompose the first integral of (5.9) into

$$\left(\int_{-\infty}^{\sigma_1^{\epsilon_1 \epsilon_2}} + \int_{\sigma_1^{\epsilon_1 \epsilon_2}}^{\sigma_2^{\epsilon_1 \epsilon_2}} + \int_{\sigma_2^{\epsilon_1 \epsilon_2}}^{+\infty} \right) \left((\rho^{\epsilon_1 \epsilon_2} - \epsilon_1)(u^{\epsilon_1 \epsilon_2})^2 + \frac{\epsilon_2 (\rho^{\epsilon_1 \epsilon_2})^\gamma}{\gamma} - \rho^{\epsilon_1 \epsilon_2} u^{\epsilon_1 \epsilon_2} \xi \right) \phi' d\xi. \tag{5.14}$$

As $\epsilon_1, \epsilon_2 \rightarrow 0$, the limit of the sum of the first and last term of (5.14) is

$$(\sigma[\rho u] - [\rho u^2]) \phi(\sigma) + \int_{-\infty}^{+\infty} \tilde{H}(\xi - \sigma) \phi d\xi \tag{5.15}$$

with $\tilde{H}(\xi - \sigma)$ taking $\rho_- u_-$ for $\xi < \sigma$ and $\rho_+ u_+$ for $\xi > \sigma$, respectively. Applying Lemmas 5.1-5.4, one can deduce that the limit of the second term of (5.14) equals

$$\begin{aligned}
& \lim_{\epsilon_1, \epsilon_2 \rightarrow 0} \rho_*^{\epsilon_1 \epsilon_2} (\sigma_2^{\epsilon_1 \epsilon_2} - \sigma_1^{\epsilon_1 \epsilon_2}) \left(\frac{\phi(\sigma_2^{\epsilon_1 \epsilon_2}) - \phi(\sigma_1^{\epsilon_1 \epsilon_2})}{\sigma_2^{\epsilon_1 \epsilon_2} - \sigma_1^{\epsilon_1 \epsilon_2}} (u_*^{\epsilon_1 \epsilon_2})^2 + \frac{\epsilon_2 (\rho_*^{\epsilon_1 \epsilon_2})^{\gamma-1}}{\gamma} \frac{\phi(\sigma_2^{\epsilon_1 \epsilon_2}) - \phi(\sigma_1^{\epsilon_1 \epsilon_2})}{\sigma_2^{\epsilon_1 \epsilon_2} - \sigma_1^{\epsilon_1 \epsilon_2}} \right. \\
&\quad \left. - \frac{\sigma_2^{\epsilon_1 \epsilon_2} \phi(\sigma_2^{\epsilon_1 \epsilon_2}) - \sigma_1^{\epsilon_1 \epsilon_2} \phi(\sigma_1^{\epsilon_1 \epsilon_2})}{\sigma_2^{\epsilon_1 \epsilon_2} - \sigma_1^{\epsilon_1 \epsilon_2}} u_*^{\epsilon_1 \epsilon_2} + \frac{u_*^{\epsilon_1 \epsilon_2}}{\sigma_2^{\epsilon_1 \epsilon_2} - \sigma_1^{\epsilon_1 \epsilon_2}} \int_{\sigma_1^{\epsilon_1 \epsilon_2}}^{\sigma_2^{\epsilon_1 \epsilon_2}} \phi d\xi \right) \\
&\quad - \lim_{\epsilon_1, \epsilon_2 \rightarrow 0} \epsilon_1 (u_*^{\epsilon_1 \epsilon_2})^2 \left(\phi(\sigma_2^{\epsilon_1 \epsilon_2}) - \phi(\sigma_1^{\epsilon_1 \epsilon_2}) \right) = 0.
\end{aligned} \tag{5.16}$$

Thus, it follows from (5.9) that

$$\lim_{\epsilon_1, \epsilon_2 \rightarrow 0} \int_{-\infty}^{+\infty} \rho^{\epsilon_1 \epsilon_2} u^{\epsilon_1 \epsilon_2} \phi d\xi = (\sigma[\rho u] - [\rho u^2]) \phi(\sigma) + \int_{-\infty}^{+\infty} \tilde{H}(\xi - \sigma) \phi d\xi. \tag{5.17}$$

(iii). Finally, we analyze the limit of $\rho^{\epsilon_1 \epsilon_2}$ and $\rho^{\epsilon_1 \epsilon_2} u^{\epsilon_1 \epsilon_2}$ by tracking the time-dependence of the weights of the δ -measures as $\epsilon_1, \epsilon_2 \rightarrow 0$.

Taking (5.13) into account, we have for any $\psi \in C_0^\infty(R \times R^+)$

$$\begin{aligned}
& \lim_{\epsilon_1, \epsilon_2 \rightarrow 0} \int_0^{+\infty} \int_{-\infty}^{+\infty} \rho^{\epsilon_1 \epsilon_2}(x/t) \psi(x, t) dx dt \\
&= \lim_{\epsilon_1, \epsilon_2 \rightarrow 0} \int_0^{+\infty} t \left(\int_{-\infty}^{+\infty} \rho^{\epsilon_1 \epsilon_2}(\xi) \psi(\xi t, t) d\xi \right) dt \\
&= \int_0^{+\infty} (\sigma[\rho] - [\rho u]) t \psi(\sigma t, t) dt + \int_0^{+\infty} \int_{-\infty}^{+\infty} H(x - \sigma t) \psi(x, t) dx dt,
\end{aligned} \tag{5.18}$$

in which, by the definition (2.2), we get

$$\int_0^{+\infty} (\sigma[\rho] - [\rho u]) t \psi(\sigma t, t) dt = \left\langle w_1(\cdot) \delta_S, \psi(\cdot, \cdot) \right\rangle \tag{5.19}$$

with $w_1(t) = \frac{t}{\sqrt{1 + \sigma^2}} (\sigma[\rho] - [\rho u])$. Similarly, one can show that

$$\begin{aligned}
& \lim_{\epsilon_1, \epsilon_2 \rightarrow 0} \int_0^{+\infty} \int_{-\infty}^{+\infty} \rho^{\epsilon_1 \epsilon_2} u^{\epsilon_1 \epsilon_2}(x/t) \psi(x, t) dx dt \\
&= \left\langle w_2(\cdot) \delta_S, \psi(\cdot, \cdot) \right\rangle + \int_0^{+\infty} \int_{-\infty}^{+\infty} \tilde{H}(x - \sigma t) \psi(x, t) dx dt
\end{aligned} \tag{5.20}$$

with $w_2(t) = \frac{t}{\sqrt{1 + \sigma^2}} (\sigma[\rho u] - [\rho u^2])$.

The proof of Theorem 5.5 is finished. \blacksquare

6 Formation of vacuum states for the system (1.5)

In this section, we study the limit of the solutions of (1.5) and (1.7) in the case $(\rho_+, u_+) \in \overleftarrow{R} \overrightarrow{R}(\rho_-, u_-)$ with $u_- < u_+$ as $\epsilon_1, \epsilon_2 \rightarrow 0$.

According to Section 4, one can get that, on the backward centred rarefaction wave, the solution satisfies

$$\begin{cases} \xi = u^{\epsilon_1 \epsilon_2} - \sqrt{\epsilon_2 (\rho^{\epsilon_1 \epsilon_2})^{\gamma-2} (\rho^{\epsilon_1 \epsilon_2} - 2\epsilon_1)}, \\ u_- - \sqrt{\epsilon_2 \rho_-^{\gamma-2} (\rho_- - 2\epsilon_1)} < \xi < u_*^{\epsilon_1 \epsilon_2} - \sqrt{\epsilon_2 (\rho_*^{\epsilon_1 \epsilon_2})^{\gamma-2} (\rho_*^{\epsilon_1 \epsilon_2} - 2\epsilon_1)}, \quad \rho_*^{\epsilon_1 \epsilon_2} < \rho_-, \end{cases} \tag{6.1}$$

and, on the forward centred rarefaction wave,

$$\begin{cases} \xi = u^{\epsilon_1 \epsilon_2} + \sqrt{\epsilon_2 (\rho^{\epsilon_1 \epsilon_2})^{\gamma-2} (\rho^{\epsilon_1 \epsilon_2} - 2\epsilon_1)}, \\ u_*^{\epsilon_1 \epsilon_2} + \sqrt{\epsilon_2 (\rho_*^{\epsilon_1 \epsilon_2})^{\gamma-2} (\rho_*^{\epsilon_1 \epsilon_2} - 2\epsilon_1)} < \xi < u_+ + \sqrt{\epsilon_2 \rho_+^{\gamma-2} (\rho_+ - 2\epsilon_1)}, \quad \rho_*^{\epsilon_1 \epsilon_2} < \rho_+. \end{cases} \tag{6.2}$$

Now, we can conclude the following theorem.

Theorem 6.1. *Let $u_- < u_+$. Assume $(\rho^{\epsilon_1 \epsilon_2}, u^{\epsilon_1 \epsilon_2})$ is a two-rarefaction wave solution of (1.5) and (1.7) constructed in Section 4. Then, there exist $\epsilon_0 > 0$, when $0 < \epsilon_1 < \epsilon_0$ and $0 < \epsilon_2 < \epsilon_0$, the constant density solution ($\rho = 2\epsilon_1$) appears in the solution. And as $\epsilon_1, \epsilon_2 \rightarrow 0$, the two rarefaction waves become two contact discontinuities connecting the constant states (u_\pm, ρ_\pm) and the vacuum ($\rho = 0$), which form a vacuum solution of (1.1) with the Riemann data (1.7).*

Proof. Set $\epsilon_1 = \epsilon_2 = \epsilon_0$. Since $(\rho_*^{\epsilon_1\epsilon_2}, u_*^{\epsilon_1\epsilon_2})$ is on the curve $\overleftarrow{R}(\rho_-, u_-)$, we have

$$u_*^{\epsilon_1\epsilon_2} = u_- - \int_{\rho_-}^{\rho_*^{\epsilon_1\epsilon_2}} \sqrt{\frac{\epsilon_0 s^{\gamma-2}}{s - 2\epsilon_0}} ds \leq u_- + \int_{2\epsilon_0}^{\rho_-} \sqrt{\frac{\epsilon_0 s^{\gamma-2}}{s - 2\epsilon_0}} ds = A^{\epsilon_0}. \quad (6.3)$$

When $u_- < u_+ < A^{\epsilon_0}$, there is no constant-density in the solution. That is, there exist ϵ_{01} such that $(\rho_+, u_+) \in I(\rho_-, u_-)$ when $u_- < u_+ < A^{\epsilon_{01}}$.

However, when $A^{\epsilon_0} < u_+$, the constant density solution appears, which implies that there exist ϵ_{02} such that $(\rho_+, u_+) \in V(\rho_-, u_-)$ when $A^{\epsilon_{02}} < u_+$.

Let $f(\epsilon) = \int_{2\epsilon}^{\rho_-} \sqrt{\frac{\epsilon s^{\gamma-2}}{s - 2\epsilon}} ds - u_+ + u_-$. Since the integral $\int_0^{\rho_-} \sqrt{\frac{s^{\gamma-2} + 2}{s}} ds$ is convergent, one can deduce that, thanks to M-criterion, the integral $\int_{2\epsilon}^{\rho_-} \sqrt{\frac{\epsilon s^{\gamma-2}}{s - 2\epsilon}} ds$ is uniformly convergent in ϵ , then the function $f(\epsilon)$ is continuous with respect to ϵ and $f(\epsilon_{01})f(\epsilon_{02}) < 0$. Thus, there exists $\epsilon_0 \in [\epsilon_{02}, \epsilon_{01}]$ such that $f(\epsilon_0) = 0$.

So when $0 < \epsilon_1 < \epsilon_0$ and $0 < \epsilon_2 < \epsilon_0$, the density of the intermediate state becomes a constant with

$$(\rho_*^{\epsilon_1\epsilon_2}, u_*^{\epsilon_1\epsilon_2})(\xi) = (2\epsilon_1, \xi), \quad u_1^{\epsilon_1\epsilon_2} \leq \xi \leq u_2^{\epsilon_1\epsilon_2}, \quad (6.4)$$

where

$$u_1^{\epsilon_1\epsilon_2} = u_- + \int_{2\epsilon_1}^{\rho_-} \sqrt{\frac{\epsilon_2 s^{\gamma-2}}{s - 2\epsilon_1}} ds, \quad u_2^{\epsilon_1\epsilon_2} = u_+ - \int_{2\epsilon_1}^{\rho_+} \sqrt{\frac{\epsilon_2 s^{\gamma-2}}{s - 2\epsilon_1}} ds.$$

Thus, letting $\epsilon_1, \epsilon_2 \rightarrow 0$, one can find $\lim_{\epsilon_1, \epsilon_2 \rightarrow 0} \rho_*^{\epsilon_1\epsilon_2} = 0$. Using the uniform boundedness of $\rho^{\epsilon_1\epsilon_2}$ with respect to ϵ_1 and ϵ_2 , it follows that

$$\lim_{\epsilon_1, \epsilon_2 \rightarrow 0} u_1^{\epsilon_1\epsilon_2} = u_-, \quad \lim_{\epsilon_1, \epsilon_2 \rightarrow 0} u_2^{\epsilon_1\epsilon_2} = u_+,$$

$$\lim_{\epsilon_1, \epsilon_2 \rightarrow 0} u^{\epsilon_1\epsilon_2}(\xi) = \xi \quad \text{for } \xi \in (u_-, u_+).$$

In summary, the limit solution for this case can be expressed as (2.1), which is a solution of (1.1) containing two contact discontinuities $\xi = x/t = u_{\pm}$ and a vacuum state in between. This completes the proof of Theorem 6.1. \blacksquare

Remark. The processes of formation of delta shock waves and vacuum states can be examined with some numerical results as ϵ_1 and ϵ_2 decrease. The numerical simulations will be presented in the version for publication.

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